Computing moment polytopes and applications

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Work with:

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First : what are moment polytopes ?

Moment polytopes: representation theory

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$$\subseteq \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$$

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$$\frac{T_1^* T_1}{\mathrm{Tr}(T_1^* T_1)}$$

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$$\Delta^{\mathrm{geom}}(T) := \left\{ \left(\operatorname{spec} \mu_1(T'), \operatorname{spec} \mu_2(T'), \operatorname{spec} \mu_3(T') \right) \mid T' \in \overline{G \cdot T} \right\}$$

$$\subseteq \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$$

Theorem [Mumford-Ness]

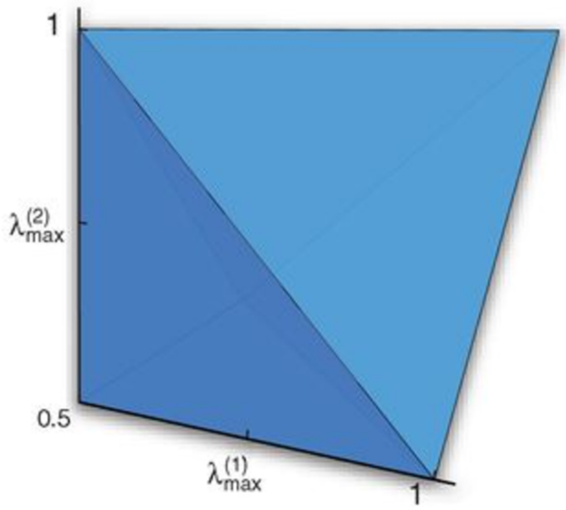
$$\Delta^{repr}(T) = \Delta^{geom}(T)$$

and it is a rational (convex compact) polytope

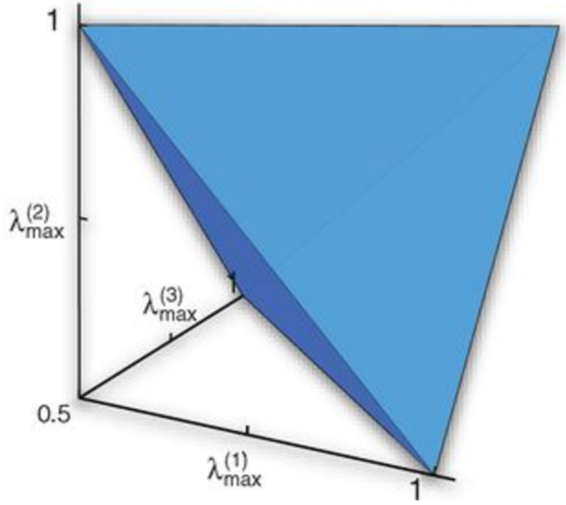
We denote it with $\Delta(T)$.

Examples

$$\Delta\left(\sum_{i=1}^{2} e_i\otimes e_i\otimes e_i\right) = \operatorname{conv}\left\{\begin{array}{l} (\overbrace{1,0,}^{\mathbb{R}^2}\ \overbrace{1,0,}^{\mathbb{R}^2}\ \overbrace{1,0}^{\mathbb{R}^2}) \\ (\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\ (1,0,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ 1,0,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ 1,0) \end{array}\right\}$$



$$\Delta\left(\begin{array}{l} \ \ e_1\otimes e_1\otimes e_2 \\ +e_1\otimes e_2\otimes e_1 \\ +e_2\otimes e_1\otimes e_1 \end{array}\right) = \operatorname{conv}\left\{\begin{array}{l} (1,0,\ 1,0,\ 1,0) \\ (1,0,\ \tfrac12,\tfrac12,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ 1,0,\ \tfrac12,\tfrac12) \\ (\tfrac12,\tfrac12,\ \tfrac12,\tfrac12,\ 1,0) \end{array}\right\}$$



W, slices:

$$\begin{matrix}0 & 1\\ 1 & 0\end{matrix} \qquad \begin{matrix}1 & 0\\ 0 & 0\end{matrix}$$

Why care?

- Fundamental objects in quantum information

  [Walter, Doran, Gross, Christandl — Entanglement polytopes: Multiparticle information from Single-Particle Information, 2013]

- Applications in matrix multiplication complexity: quantum functionals

  [Bürgisser, Ikenmeyer — Geometric complexity theory and tensor rank, 2011]

  [Christandl, Vrana, Zuiddam — Universal points in the asymptotic spectrum of tensors, 2018]

- Examples of general moment polytopes, central objects in scaling problems

  [Bürgisser, Franks, Garg, Oliveira, Walter, Wigderson — Towards a theory of non-commutative optimisation (...), 2019]

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- Only known completely for $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$
  + some sporadic examples

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Franz' description

Example:

$$\mathrm{supp}(W) = \mathrm{supp}\begin{pmatrix} e_1 \otimes e_1 \otimes e_2 \\ + e_1 \otimes e_2 \otimes e_1 \\ + e_2 \otimes e_1 \otimes e_1 \end{pmatrix} = \left\{ \begin{matrix} (1,0\,,\,1,0\,,\,0,1), \\ (1,0\,,\,0,1\,,\,1,0), \\ (0,1\,,\,1,0\,,\,1,0) \end{matrix} \right\}$$

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where $T' \in GL \cdot T$ is generic

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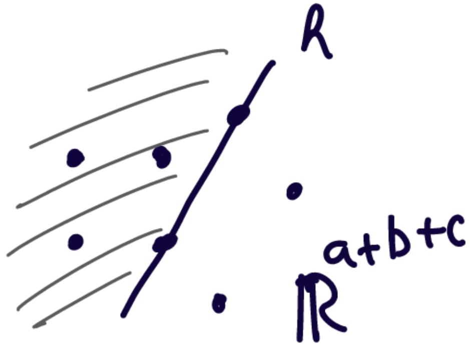

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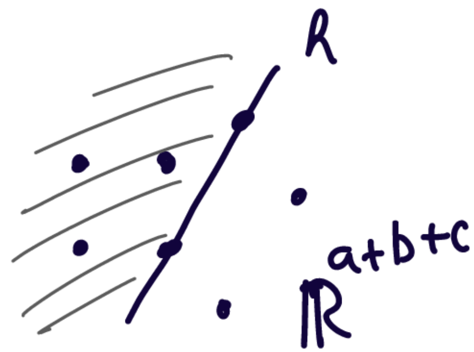
Lemma (informal) $\Delta(T)$ can be defined using inequalities R that each satisfy:

(1) R is tight on $a+b+c-3$ elements of $\{(e_i, e_j, e_k)\}_{i,j,k}$

(2)



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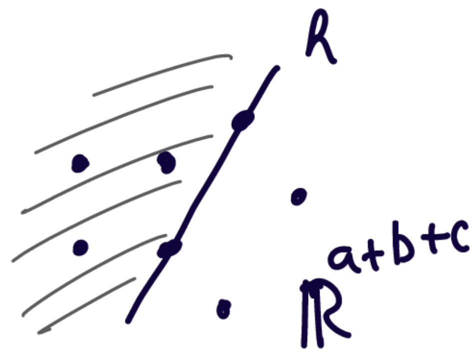

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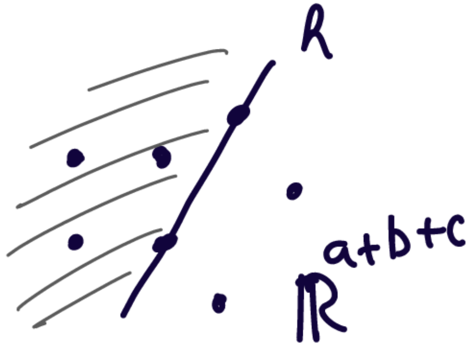

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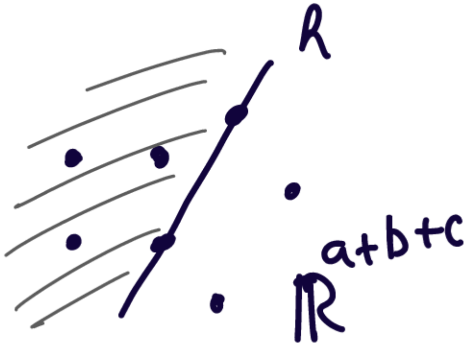

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	Inequalities					Vertices	Runtimes	
Tensor	All	Not generic	Maxranks	Attainable	Final	Final	$\mathbb{Q}$	$\mathbb{F}_q$
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$\langle 4 \rangle$	8109383	7139405	1102518	0	270	328	-	20:10
$\mathrm{MM}_{2,2,2}$	8109383	7139405	1102518	1227	129	181	-	3:06

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still going after 10 hours

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We know all moment polytopes in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$

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Results: $\Delta(MM_n)$ is not maximal

2x2 matrix multiplication

Our algorithm gave (probabilistically) that

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